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The decay of free motion of a floating body: force coefficients at large complex frequencies

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The results of Ursell (1964) are confirmed by proving that there are no additional contributions to Ursell's integral from singularities of the integrand at infinity. The method consists of proving that the asymptotic expansion of a force coefficient $\Lambda(N)$ is uniformly valid in a finite sector of the complex N-plane. This in turn requires that the kernel of an integral equation remains small in this sector.

1. Introduction

Ursell (1964) considers the problem of a floating circular cylinder which is displaced vertically, held until the water is again at rest, and then released. The problem is solved by a Fourier transform and the result expressed as a Fourier integral with a force coefficient $\Lambda(u)$ in the integrand. The behaviour of the solution depends on the singularities of this function. Ursell shows that $\Lambda(u)$ has a branch point at $u = 0$, and gives a solution to the problem on the assumption that $\Lambda(u)$ and its first few derivatives do not oscillate rapidly at infinity. The purpose of the present paper is to establish that this assumption is true. The method we use is that indicated by Ursell of showing that $\Lambda(u)$ has the same asymptotic expansion for large $|u|$ throughout some finite sector of the complex u-plane, $-\epsilon \leqslant \arg u \leqslant \epsilon$. The path of the Fourier integral can then be deformed into the lower half of the u-plane for large $|u|$ and the resulting contribution is exponentially small.

The asymptotic expansion of $\Lambda(u)$ for large real u has been found in an earlier paper (Ursell 1953), from the solution of an integral equation. The method of solution of the integral equation depends on the kernel being small for large $N = u^2$, so that an iterative procedure can be used. The integral equation method is much improved in a later paper (Ursell 1961), and for the most part the present work follows this paper. Ursell's papers (1953, 1961 and 1964) will be referred to as U 1, U 2 and U 3 respectively.

It is easily seen from U 1 that the force coefficient $\Lambda(N)$ will have a uniform asymptotic expansion for $-\epsilon \leqslant \arg N \leqslant \epsilon$ if the kernel of the integral equation from which it is derived remains small for complex N in this sector, since then the solution of the integral equations will be unchanged. Thus the problem of the present paper reduces to showing that the integral equation of U 1 can be replaced by a modified equation in which the kernel does remain small in this sector.

2. The kernel of the integral equation

The integral equation of U 1 comes from consideration of the forced motion produced by the vertical oscillations of a circular cylinder. The cylinder is of radius a, with the mean position of its axis in the undisturbed free surface, and the oscillation is such that the whole motion is proportional to $e^{-i\sigma t}$ and sufficiently small to allow linearization of the boundary conditions. By using Green's theorem Ursell finds an integral equation, U 1 (3.3), for the velocity potential $\phi(\alpha)$, $-\frac{1}{2}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$. The kernel involves a derivative with respect to r, where $x = r\sin\theta$, $y = r\cos\theta$, of the function

$$G_1(x,y;\xi,\eta) = -2\int_0^\infty \frac{\exp[-k(y+\eta)]\cos k(x-\xi)}{k-K}\,dk, \tag{1}$$

in which the path of integration is indented to pass below the pole at $k = K$. Here $Ka = \sigma^2 a/g = N$; K, N are large. Our object is to show that the kernel remains small when K has an imaginary part $K = K_1 + iK_2$; $N = N_1 + iN_2$; K_1, $N_1 > 0$. As we intend to follow the methods of U 2 we use the fact that $\phi(\alpha)$ is even and rewrite the integral in Ursell's equation with limits $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

As it stands the iterative procedure cannot be used even when K is real. This is because when the points $(a\sin\theta,\, a\cos\theta)$ and $(a\sin\alpha,\, a\cos\alpha)$ are on opposite faces of the cylinder the kernel is no longer small. Physically waves are generated by a simple oscillating source at $(a\sin\alpha,\, a\cos\alpha)$ and the large terms represent waves which have travelled through the centre of the cylinder; this is, of course, impossible in the real problem. This difficulty is removed in U 2 by modifying the original Green's function by subtracting from it suitable combinations of simple source and doublet solutions $S(Kx, Ky)$ and $D(Kx, Ky)$. In fact D is not strictly needed in our problem and only S terms are used in U 1, but we prefer to retain the form used in U 2, where S and D were derived in appendix 1.

Following U 2 (pp. 644–5) we find that the wave terms are cancelled if G_1 is replaced by

$$\mathfrak{g}_1 = G_1 - \tfrac{1}{2}(S + i\,\mathrm{sgn}\,\alpha D)\exp\{-N\cos\alpha + iN|\sin\alpha|\}, \tag{2}$$

where $\mathrm{sgn}\,\alpha = \pm 1$ according as $\alpha \gtrless 0$. Note that our θ, α are defined as in U 1, and that 'sin' and 'cos' in U 2 are replaced by 'cos' and 'sin' here.

We now write the cosine in G_1 as a sum of exponentials and rotate the contours in the two integrals to obtain integrals along the positive and negative halves of the imaginary k-axis respectively, giving

$$G_1(x,y;\xi,\eta) = -2\pi i\exp[-K(y+\eta) + iK|x-\xi|] - 2F(\hat{K}; |K|(y+\eta), |K|\,|x-\xi|), \tag{3}$$

where

$$F(\hat{K};\lambda,\mu) = \tfrac{1}{2}i\int_0^\infty \frac{\exp[-v(\mu - i\lambda)]}{\hat{K}+iv}\,dv - \tfrac{1}{2}i\int_0^\infty \frac{\exp[-v(\mu+i\lambda)]}{\hat{K}-iv}\,dv, \tag{4}$$

and $\hat{K} = K/|K|$. This is following U 2, appendix A, but the definition of the function F is modified to allow for K being complex. When K is real, $\hat{K} = 1$ and F is the same as in U2 (A1.8). A summation which appears in the definition of S comes from the asymptotic expansion of F for large $\lambda^2+\mu^2$ and is easily seen to be unchanged, when K is complex, from that given in U2. Now we can write down the new kernel as $\mathfrak{K}^{(1)}+\mathfrak{K}^{(2)}$, where

$$\mathfrak{K}^{(1)} = -\left[a\frac{\partial}{\partial r}\{2F(\hat{K};|K|(r\cos\theta+a\cos\alpha),|K|\,|r\sin\theta-a\sin\alpha|)\}\right]_{r=a}; \tag{5}$$

$$\mathfrak{K}^{(2)} = P\exp[iN|\sin\theta-\sin\alpha|]+Q\exp[iN\{|\sin\theta|+|\sin\alpha|\}]+R\exp[iN|\sin\alpha|]; \tag{6}$$

$$P = 2\pi N\exp[-N(\cos\theta+\cos\alpha)](i\cos\theta+\sin\theta\,\mathrm{sgn}(\sin\theta-\sin\alpha)); \tag{7}$$

$$Q = \pi N\exp[-N(\cos\theta+\cos\alpha)](-1+\mathrm{sgn}\,\theta\,\mathrm{sgn}\,\alpha)(i\cos\theta+|\sin\theta|); \tag{8}$$

$$R = \left[\exp[-N\cos\alpha]\,a\frac{\partial}{\partial r}\left\{\left[F(\hat{K};|K|y;|K|\,|x|)\right.\right.\right.$$
$$\left.\left.\left.-\tfrac{1}{2}i\sum_{m=0}^{s}\frac{m!}{K^{m+1}}\left(\frac{(-i)^m}{(y-i|x|)^{m+1}}-\frac{i^m}{(y+i|x|)^{m+1}}\right)\right]+\frac{i\,\mathrm{sgn}\,\alpha}{K}\frac{\partial}{\partial x}\left[\quad\right]\right\}\right]_{r=a}. \tag{9}$$

The summations in R come from S and D, but we have retained the series form of U2 (A1.9) rather than that of U2 (A1.10). The upper limit of summation, s, can take any finite even value, although the series is in fact divergent; it must be even in order to satisfy the surface boundary condition. Note that, when $\mathrm{sgn}\,\theta = -\mathrm{sgn}\,\alpha$, the P and Q wave terms cancel, and when $\mathrm{sgn}\,\theta = \mathrm{sgn}\,\alpha$, $Q = 0$.

3. Bounds for the kernel

To find bounds for this kernel when N is large we shall consider three cases separately: (i) $\mathfrak{K}^{(2)}$ when $\mathrm{sgn}\,\theta = -\mathrm{sgn}\,\alpha$; (ii) $\mathfrak{K}^{(2)}$ when $\mathrm{sgn}\,\theta = \mathrm{sgn}\,\alpha$; and (iii) $\mathfrak{K}^{(1)}$.

(i) Here $\mathfrak{K}^{(2)}$ is simply the R term. In the series appearing in (9) we have yet to specify the value of the upper limit s. As just remarked the series is derived from the asymptotic expansion of the function $F(\hat{K};\lambda,\mu)$ for large $\lambda^2+\mu^2$. The method used is Watson's lemma (Watson 1944, p. 236) which involves expansion inside the integral sign and integrating term by term. In this section to find a bound for R for large $|K|$ we rewrite the summations inside the integral. Thus

$$R = \left[\exp[-N\cos\alpha]\,a\frac{\partial}{\partial r}\left\{\left[\tfrac{1}{2}i\int_0^\infty\exp[-v|K|(|x|-iy)]\right.\right.\right.$$
$$\times\left\{\frac{1}{\hat{K}+iv}-\frac{1}{\hat{K}}\sum_{m=0}^{s}\left(\frac{-iv}{\hat{K}}\right)^m\right\}dv$$
$$\left.\left.\left.-\tfrac{1}{2}i\int_0^\infty\exp[-v|K|(|x|+iy)]\left\{\frac{1}{\hat{K}-iv}-\frac{1}{\hat{K}}\sum_{m=0}^{s}\left(\frac{iv}{\hat{K}}\right)^m\right\}dv\right]+\frac{i\,\mathrm{sgn}\,\alpha}{K}\frac{\partial}{\partial x}\left[\quad\right]\right\}\right]_{r=a}. \tag{10}$$

The summations here are simple geometric progressions for which we can write down the sum in closed form, so the first integral is

$$\int_0^\infty\exp[-v|K|(|x|-iy)]\left(\frac{-iv}{\hat{K}}\right)^s\frac{1}{\hat{K}+iv}\,dv \tag{11}$$

$$= \left(\frac{-i}{\hat{K}}\right)^s\int_0^\infty\frac{\exp[-v|K|(|x|-iy)+s\ln v]}{\hat{K}+iv}\,dv. \tag{12}$$

The second integral is obtained from this by changing the sign of i except in $\hat{K}$. The integral can now be approximated for large $|K|$ by the method of steepest descents, allowing the possibility that s may be $O(|K|)$ since we have not yet fixed its value. It turns out that the path of steepest descent is a straight line through $v = 0$ making an angle $\frac{1}{2}\pi - |\theta|$ with the real axis. (If in diverting the contour the pole at $iv = -\hat{K}$ is encountered the contour must be indented to pass the pole on the side nearer to the real axis, since its contribution has already been added in.) Standard application of the method then gives as the first term in the asymptotic expansion of (12)

$$(2\pi s)^{\frac{1}{2}} \left(\frac{s}{Kr}\right)^s \frac{\exp\left[i(\frac{1}{2}\pi - (s+1)|\theta|) - s\right]}{(Kr - s\exp[-i|\theta|])}. \tag{13}$$

Higher terms can also be calculated, but all contain the factor e^{-s}. Similarly, the second integral in (10) also has this factor in its asymptotic expansion, and the further differentiations involved in calculating R leave the e^{-s} unchanged. Thus with $N = N_1 + iN_2$ the $\mathfrak{K}^{(2)}$ part of the kernel contains the factor

$$\exp\left[-s - N_1\cos\alpha - N_2|\sin\alpha| + i(\text{real function})\right] \tag{14}$$

and no other exponentials. If s were small and $N_2 < 0$ this would be very large near to $\alpha = \pm\frac{1}{2}\pi$. However, we have allowed s to be $O(|N|)$ and we are at liberty to choose its actual value; if we choose $s = [|N|] + 1$, or $[|N|] + 2$ where $[|N|]$ is the greatest integer $\leqslant |N|$ and the 1 or 2 is taken so as to make s even, $\mathfrak{K}^{(2)}$ is exponentially small for all N.

(ii) When $\operatorname{sgn}\theta = \operatorname{sgn}\alpha$, the R term can be made exponentially small in the same way, $Q = 0$, and we are left with the P term. The exponential factor in this term can be expressed as

$$\exp\left[-N_1\, 2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}\left(1 + \tan\epsilon\left|\tan\frac{\theta-\alpha}{2}\right|\right) + i(\text{real function})\right], \tag{15}$$

where we have written $N_2 = N_1\tan\epsilon$. But $|\tan\frac{1}{2}(\theta-\alpha)| \leqslant 1$ since $|\theta-\alpha| \leqslant \frac{1}{2}\pi$ here, and therefore, if $-1 < \tan\epsilon < \infty$, i.e. $-\frac{1}{4}\pi < \epsilon < \frac{1}{2}\pi$, this factor is exponentially small for large N_1 (>0) and hence for large $|N|$. Thus if $\epsilon < 0$ we must specify $-N_2 < N_1$, but with this restriction $\mathfrak{K}^{(2)}$ is exponentially small.

(iii) To find bounds for $\mathfrak{K}^{(1)}$ we follow U2, appendix 2. Writing $\theta^* = \frac{1}{2}\pi - \theta$, $\alpha^* = \frac{1}{2}\pi - \alpha$, so that θ^*, α^* are the θ, α of U2, the theory follows through without modifications except that the known result U2 (A2.18) has to be extended to the integral

$$I = \int_0^\infty \frac{e^{-kz}}{k-k'}\,dk, \tag{16}$$

where $k' = 1 + i\tan\epsilon$, $\epsilon = \arg K$, and the path passes below the pole. Substituting $k = k'l$ and rotating the contour back to the real axis, we have

$$I = \int_0^\infty \frac{\exp\left[-(k'z)\,l\right]}{l-1}\,dl + \lim_{R\to\infty}\int_0^{-\epsilon} \frac{\exp\left[-(k'z)R\exp(i\chi)\right]}{Re^{i\chi}-1}\, iRe^{i\chi}d\chi \tag{17}$$

$$= \text{principal value}\int_0^\infty \frac{\exp\left[-(k'z)\,l\right]}{l-1}\,dl + \pi i\exp\left[-k'z\right] \tag{18}$$

provided that

$$|\epsilon + \arg z + \chi| < \tfrac{1}{2}\pi \quad \text{for} \quad \begin{Bmatrix} -\epsilon < \chi < 0, \ \epsilon > 0 \\ 0 < \chi < -\epsilon, \ \epsilon < 0 \end{Bmatrix}. \tag{19}$$

Since we have already specified $\epsilon > -\frac{1}{4}\pi$ we can without any real loss put $|\epsilon| < \frac{1}{4}\pi$. Then this condition will be satisfied if $|\arg z| \leqslant \frac{1}{4}\pi$. With this restriction

$$I = -\exp[-(k'z)]\ln(k'z) + (\text{a regular function of } k'z), \tag{20}$$

following U 2, (A 2.18), and the remainder of this section follows from U 2.

Thus the kernel remains bounded when N is complex provided that

$$|\arg N| < \tfrac{1}{4}\pi.$$

The same bounds are available as those used in U 2, and the iterative solution of the integral equation may proceed as before. It follows that the asymptotic expansion of $\Lambda(N)$ is uniformly valid for $|\arg N| < \frac{1}{4}\pi$ and hence the results of U 3 are confirmed.

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